Position-momentum local realism violation of the Hardy type

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We show that it is, in principle, possible to perform local realism violating experiments of the Hardy type in which only position and momentum measurements are made on two particles emanating from a common source. In the optical domain, homodyne detection of the in-phase and out-of-phase amplitude components of an electromagnetic field is analogous to position and momentum measurement. Hence, local realism violations of the Hardy type are possible in optical systems employing only homodyne detection.

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As an example to support their contention that quantum mechanics is incomplete Einstein, Podolsky, and Rosen (EPR) [1] exhibited a quantum mechanical wave function, describing two particles emitted from a common source, in which the positions and the momenta of the two particles were strongly correlated. This wave function described the situation in which the measurement of the position of one of the particles would allow one to predict with complete certainty the position of the other particle and the measurement of the momentum of one of the particles would allow one to predict with complete certainty the momentum of the other particle. Because of these strong correlations even when the particles were well-separated it was argued that each of the particles must have a definite position and a definite momentum even though a quantum mechanical wave function does not simultaneously ascribe a definite position and a definite momentum to a particle. Therefore, it was argued that quantum mechanics is incomplete. It was hoped that in the future a complete theory could be devised in which a definite position and definite momentum would be ascribed to each particle. In 1992 the EPR Gedanken experiment was actually carried out [2] as a quantum optics experiment in which electromagnetic field analogues of position and momentum were measured on correlated photon states generated by parametric down-conversion [3,4]. The analogues of the position and momentum were the two quadrature amplitudes of the electromagnetic field measured via homodyne detectors [5–7]. A quantum mechanical state having the properties of the state employed by EPR had been realized.

However, since the work of Bell [8] it has been known that a complete theory, of the type EPR hoped for, capable of making the same predictions as quantum mechanics, does not exist [9]. A variety of experiments, referred to as local realism violating experiments, have been proposed and performed, demonstrating that quantum mechanics is inherently at odds with classical notions about how effects propagate. Most striking among the proposals are the "one event" local realism violating experiments devised by Greenberger Horne and Zeilinger (GHZ) [10–12] and by Hardy [13–15]. The Bell, GHZ, and Hardy experiments that have been proposed generally measure spin components or count particles, i.e., they employ observables that have a discrete spectrum. There are, however, some examples in which continuous observables or a mixture of discrete and continuous observables have been employed [16–19]. In fact, Bell [16] showed that position and momentum measurements on a pair of particles in a state for which the Wigner function has negative regions can give rise to local realism violating effects of the Clauser, Holt, Horne, and Shimony type [20]. Here we show that local realism violating effects of the Hardy type can be obtained through position and momentum measurements on a pair of particles prepared in the appropriate state. Given that homodyne detection measurements of the two quadrature amplitude components of an electromagnetic field provide an optical analogue to position and momentum measurements, an optical experiment exhibiting local realism violations of the Hardy type can be devised, provided the appropriate entangled state can be generated.

A local realism constraint on the positions and momenta measured for a pair of particles emitted from a common source can be arrived at by regarding the detectors as responding to messages emitted by the source [11,18]. The source does not know, ahead of time, whether a position or a momentum measurement will be performed by a given detector. Hence, the instruction set emitted by the source must tell the detectors what to do in either case. The instruction sets are conveniently labeled via the array $(\alpha_{x1}, \alpha_{x2}; \alpha_{p1}, \alpha_{p2})$ where α_{xi} and α_{pi} are members of the set $\{+, -\}$. Here α_{xi} denotes whether detector i, measuring the position x_i of particle i, will report the position to be positive ($\alpha_{xi} = +$) or negative ($\alpha_{xi} = -$). Similarly, α_{pi} denotes whether detector i, measuring the momentum p_i of particle i, will report the momentum to be positive ($\alpha_{pi} = +$) or negative ($\alpha_{pi} = -$). The probability that a message of the form $(\alpha_{x1}, \alpha_{x2}; \alpha_{p1}, \alpha_{p2})$ will be denoted as $P(\alpha_{x1}, \alpha_{x2}; \alpha_{p1}, \alpha_{p2})$. Let $P_{\beta_1\beta_2}(\alpha_{\beta_1 1}, \alpha_{\beta_2 2})$, where $\beta_i \in \{x, p\}$, is the probability that detector 1 measuring β_1 reports $\alpha_{\beta_1 1}$ while detector 2 measuring β_2 reports $\alpha_{\beta_2 2}$. For example, $P_{xp}(+, -)$ denotes the joint probability that detector 1 measuring position will report a positive position while detector 2 measuring momentum will report a negative momentum. In terms of the message probabilities, $P_{pp}(-, -)$ is given by

$$P_{pp}(-, -) = P(+, +; -, -) + P(+, -; -, -)$$

$$+ P(-,+;-,-) + P(-,-;-,-) \ . \tag{1}$$

The joint probabilities provide the following bounds on the message probabilities

$$P(+,+;-,-) \le P_{xx}(+,+) \ , \tag{2}$$

$$P(+,-;-,-) \le P_{px}(-,-) \ , \tag{3}$$

$$P(-,+;-,-) \le P_{xp}(-,-) \ , \tag{4}$$

and

$$P(-,-;-,-) \le \min\{P_{xp}(-,-), P_{px}(-,-)\} \ . \tag{5}$$

Applying these inequalities to Eq. (1) yields the following local realism constraint on the joint probabilities:

$$\begin{aligned} P_{pp}(-,-) \le P_{xx}(+,+) + P_{px}(-,-) + P_{xp}(-,-) \\ + \ \min\{P_{xp}(-,-), P_{px}(-,-)\} \ . \end{aligned} \tag{6}$$

If it is rigorously known that the probabilities on the right-hand side of the inequality (6) are all zero,

$$P_{xx}(+,+) = P_{xp}(-,-) = P_{px}(-,-) = 0 \ , \tag{7}$$

then it follows, according to local realism, that $P_{pp}(-,-)$ is rigorously zero. Thus, the appearance of a single event in which both particles have negative momentum would violate local realism. This situation, referred to as "one event" local realism violation, of course, cannot be achieved in practice because with a finite amount of data or the presence of spurious events it is impossible to rigorously demonstrate, experimentally, that Eq. (7) holds for a given physical system. Nevertheless, if the spurious event rate is sufficiently small, it is possible to demonstrate to a high degree of certainty with a finite amount of data that the inequality Eq. (6) is violated.

It is shown here how a wave function can be constructed that satisfies Eq. (7) and for which the joint probability on the left-hand side of (5) is nonzero

$$P_{pp}(-,-) \geq 0 \ . \tag{8}$$

Let the wave function be denoted by $\psi_{\beta_1\beta_2}$, depending on the representation. For example, ψ_{xp} is the wave function in the representation in which the position coordinate of particle 1 and the momentum coordinate of particle 2 are employed. Eq. (7) imposes the following conditions on the wave function:

$$\psi_{xx}(x_1, x_2) = 0 \text{ when } x_1 \geq 0 \text{ and } x_2 \geq 0 \ , \tag{9}$$

$$\psi_{px}(p_1, x_2) = 0 \text{ when } p_1 \leq 0 \text{ and } x_2 \leq 0 \ , \tag{10}$$

and

$$\psi_{xp}(x_1, p_2) = 0 \text{ when } x_1 \leq 0 \text{ and } p_2 \leq 0 \ . \tag{11}$$

A wave function satisfying these conditions can be constructed as follows: Let $g(p_1, p_2)$ be a function that is nonzero only when p_1 and p_2 are positive, i.e.,

$$g(p_1, p_2) = 0 \text{ if } p_1 \leq 0 \text{ or } p_2 \leq 0 \ . \tag{12}$$

Its Fourier transform, denoted by $f(x_1, x_2)$, is

$$f(x_1, x_2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(p_1 x_1 + p_2 x_2)} g(p_1, p_2) \; dp_1 dp_2 \; . \tag{13}$$

The wave function ψ_{xx} is then given by

$$\psi_{xx}(x_1, x_2) = N[1 - \theta(x_1)\theta(x_2)] f(x_1, x_2) \tag{14}$$

where $\theta(x)$ is the Heaviside function defined by

$$\theta(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} \tag{15}$$

and N is the normalization coefficient chosen so that $\psi_{xx}(x_1, x_2)$ is normalized:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\psi_{xx}(x_1, x_2)|^2 = 1 \; . \tag{16}$$

Eq. (9) is enforced by the factor in square brackets appearing in Eq. (14). That Eq. (10) is also satisfied is now demonstrated. ψ_{px} is a Fourier transform of ψ_{xx}:

$$\psi_{px}(p_1, x_2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ip_1 x_1} \psi_{xx}(x_1, x_2) \; dx_1 \; . \tag{17}$$

But, from Eq. (14) this reduces to

$$\psi_{px}(p_1, x_2) = \frac{N}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ip_1 x_1} f(x_1, x_2) \; dx_1 \tag{18}$$

when $x_2 \leq 0$. Substituting Eq. (13) into this and carrying out the x_1 integration followed by a momentum integration yields

$$\psi_{px}(p_1,x_2) = \frac{N}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ip_2x_2} g(p_1,p_2) dp_2 \tag{19}$$

when $x_2 \leq 0$. It is evident from Eq. (12) that the right-hand side of Eq. (19) is zero when $p_1 \leq 0$, that is, Eq. (10) is satisfied. A similar argument shows that the wave function of Eq. (14) also satisfies Eq. (11). Transforming Eq. (14) into the momentum representation for both particles yields, keeping Eq. (12) in mind,

$$\psi_{pp}(p_1,p_2) = -\frac{N}{2\pi} \int_0^{\infty} \int_0^{\infty} e^{-i(p_1x_1+p_2x_2)} f(x_1,x_2)\ dx_1 dx_2$$
$$\text{when } p_1 \leq 0 \text{ and } p_2 \leq 0\ . \tag{20}$$

$\psi_{pp}(p_1,p_2)$ evaluated over this range is what is needed to compute $P_{pp}(-,-)$:

$$P_{pp}(-,-) = \int_{-\infty}^{0} \int_{-\infty}^{0} |\psi_{pp}(p_1,p_2)|^2 dp_1 dp_2\ . \tag{21}$$

If $\psi_{pp}(p_1,p_2) \neq 0$ over some region in the domain ($p_1 < 0$ and $p_2 < 0$), then a wave function has been constructed that violates the local realism condition Eq. (6).

We now specialize to the case when $g(p_1,p_2)$ factorizes as follows:

$$g(p_1,p_2) = g(p_1)g(p_2) \tag{22}$$

where

$$g(p) = 0 \text{ for } p \leq 0\ . \tag{23}$$

Then $f(x_1,x_2)$ factorizes,

$$f(x_1,x_2) = f(x_1)f(x_2)\ , \tag{24}$$

where

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{ipx} g(p) dp \ . \tag{25}$$

Also, Eq. (20) reduces to

$$\psi_{pp}(p_1, p_2) = -N\psi_p(p_1)\psi_p(p_2) \text{ when } p_1 \le 0 \text{ and } p_2 \le 0 \tag{26}$$

where

$$\psi_p(p) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-ipx} f(x) dx \text{ when } p \le 0 \ . \tag{27}$$

Substituting Eq. (26) into Eq. (21) yields

$$P_{pp}(-,-) = N^2 \left[\int_{-\infty}^0 |\psi_p(p)|^2 dp \right]^2 \ . \tag{28}$$

As a specific example, let $g(p)$ be given by

$$g(p) = \begin{cases} \sqrt{2\lambda} e^{-\lambda p} & \text{for } p > 0 \\ 0 & \text{for } p \le 0 \end{cases} \ . \tag{29}$$

From this, using Eq. (25), one obtains

$$f(x) = i \sqrt{\frac{\lambda}{\pi}} \frac{1}{x + i\lambda} \ . \tag{30}$$

Substituting this into Eq. (14), using Eq. (24), and computing the norm, one obtains

$$N = \frac{2}{\sqrt{3}} \ . \tag{31}$$

Substituting Eq. (30) into Eq. (27) yields, for $p \le 0$,

$$\psi_p(p) = \frac{i}{\pi}\sqrt{\frac{\lambda}{2}}\left[\int_0^\infty \frac{\cos(|p|x)}{x+i\lambda}dx + i\int_0^\infty \frac{\sin(|p|x)}{x+i\lambda}dx\right] \ . \tag{32}$$

By breaking the right-hand side of this equation into real and imaginary parts and by making use of formulas given by Gradshteyn and Ryzhik [21] (section 3.723, Eqs. 1 through 4), this equation simplifies to

$$\psi_p(p) = -\frac{i}{\pi}\sqrt{\frac{\lambda}{2}}e^{\lambda|p|}\mathrm{Ei}(-\lambda|p|) \ . \tag{33}$$

From this one finds

$$\int_{-\infty}^0 |\psi_p(p)|^2 dp = \frac{1}{2\pi^2}\int_0^\infty e^{2x}\mathrm{Ei}^2(-x)dx \ . \tag{34}$$

By performing a numerical integration of this equation we have found that

$$\int_{-\infty}^0 |\psi_p(p)|^2 dp = \frac{1}{8} \tag{35}$$

to one part in 10^8. From Eqs. (28) and (31) one thus obtains

$$P_{pp}(-,-) = \frac{1}{48} \ . \tag{36}$$

Thus, for a system possessing the wave function described here, a local realism violating event in which the momenta of both particles are negative occurs at a rate of one event in 48 events.

It has been shown here that local realism violating experiments of the Hardy type are, in principle, possible in which only position and momentum measurements are performed. A means of experimentally generating the appropriate states has not been offered, so it remains to be seen whether such states can be realized in practice. In this regard we derive hope from the fact that the experiment proposed by EPR became realizable 57 years later as an optical analogue (through the development of parametric down-converters and homodyne detectors) and we take heart in the fact that state synthesis is an active topic of research [22].

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